

EXTRINSIC DIOPHANTINE APPROXIMATION ON MANIFOLDS AND FRACTALS

LIOR FISHMAN AND DAVID SIMMONS

ABSTRACT. Fix $d \in \mathbf{N}$, and let $S \subseteq \mathbb{R}^d$ be either a real-analytic manifold or the limit set of an iterated function system (for example, S could be the Cantor set or the von Koch snowflake). An *extrinsic* Diophantine approximation to a point $\mathbf{x} \in S$ is a rational point \mathbf{p}/q close to \mathbf{x} which lies *outside* of S . These approximations correspond to a question asked by K. Mahler ('84) regarding the Cantor set. Our main result is an extrinsic analogue of Dirichlet's theorem. Specifically, we prove that if S does not contain a line segment, then for every $\mathbf{x} \in S \setminus \mathbb{Q}^d$, there exists $C > 0$ such that infinitely many vectors $\mathbf{p}/q \in \mathbb{Q}^d \setminus S$ satisfy $\|\mathbf{x} - \mathbf{p}/q\| < C/q^{(d+1)/d}$. As this formula agrees with Dirichlet's theorem in \mathbb{R}^d up to a multiplicative constant, one concludes that the set of rational approximants to points in S which lie outside of S is large. Furthermore, we deduce extrinsic analogues of the Jarník–Schmidt and Khinchin theorems from known results.

1. INTRODUCTION

Fix $d \in \mathbf{N}$ and a set $S \subseteq \mathbb{R}^d$. One may divide the set of rational points into two disjoint classes: the class of rational points which lie on S , and the class of rational points which lie outside of S . Approximating points in S by rational points in S is called *intrinsic* approximation, while approximating points in S by rational points outside of S is called *extrinsic* approximation. More well-studied is the case where the approximations may come from either inside or outside S ; in this case the approximations will be called *ambient*.

We shall be particularly interested in two classes of sets: S may be either the limit set of an iterated function system or a real-analytic manifold. Of particular prominence is the Cantor set,¹ of which K. Mahler [17] asked: “How close can irrational elements of Cantor's set be approximated by rational numbers (a) In Cantor's set, and (b) By rational numbers not in Cantor's set?” In our terminology, Mahler is asking about intrinsic and extrinsic approximation on the Cantor set, respectively. For both the limit sets of iterated function systems and for manifolds, there is already literature on both intrinsic and ambient approximation; see for example [1, 4, 11, 12] and the references therein. By contrast, extrinsic approximation on algebraic varieties has been studied only briefly, in [8, Lemma 1], [9, Lemma 4.1.1], and [6, Lemma 1]. Each of these papers proved a lemma which stated that extrinsic rational approximations to points on algebraic varieties cannot be too close to the points they approximate.

In this paper, we analyze the theory of extrinsic approximation in more detail. Our main result (Theorem 1.1) is an extrinsic analogue of Dirichlet's theorem. We also describe results concerning extrinsic approximation which may be deduced from their intrinsic and ambient counterparts, namely analogues of the Jarník–Schmidt theorem and Khinchin's theorem.

Convention 1. The symbols \lesssim , \gtrsim , and \asymp will denote multiplicative asymptotics. For example, $A \lesssim_K B$ means that there exists a constant $C > 0$ (the *implied constant*), depending only on K , such that $A \leq CB$. In general, dependence of the implied constant(s) on universal objects such as the set S will be omitted from the notation.

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¹In this paper, the phrase “Cantor set” always refers to the ternary Cantor set.

1.1. An extrinsic analogue of Dirichlet's theorem. Our main theorem is as follows:

Theorem 1.1. *Fix $d \in \mathbb{N}$, and let $S \subseteq \mathbb{R}^d$ be either*

- (1) *the limit set of an iterated function system² (cf. Definition 2.10), or*
- (2) *a real-analytic manifold,*

and suppose that S does not contain a line segment. Then for all $\mathbf{x} \in S \setminus \mathbb{Q}^d$, there exists $C = C_{\mathbf{x}} > 0$ such that infinitely many $\mathbf{p}/q \in \mathbb{Q}^d \setminus S$ satisfy

$$(1.1) \quad \left\| \mathbf{x} - \frac{\mathbf{p}}{q} \right\| \leq \frac{C}{q^{1+1/d}}.$$

Here and elsewhere $\|\cdot\|$ denotes the max norm. Moreover, the function $\mathbf{x} \mapsto C_{\mathbf{x}}$ is bounded on compact sets.

We recall that (the corollary of) Dirichlet's theorem in \mathbb{R}^d states that for all $\mathbf{x} \in \mathbb{R}^d \setminus \mathbb{Q}^d$, there exist infinitely many $\mathbf{p}/q \in \mathbb{Q}^d$ satisfying (1.1) with $C = 1$. Thus Theorem 1.1 says that if S is as above, then for each $\mathbf{x} \in S \setminus \mathbb{Q}^d$ there are enough extrinsic approximations to \mathbf{x} to re-prove Dirichlet's theorem, if one is content with a constant multiplicative error term.³ This can be contrasted with the situation for intrinsic approximation, where the best theorem that one can prove using intrinsic rationals is much worse than Dirichlet's theorem [11, Theorem 4.3].

As a concrete application of Theorem 1.1, we consider the case where S is the Cantor set, thus giving an answer to the second of Mahler's questions mentioned above. Since the Cantor set is compact and does not contain a line segment, the following is an immediate corollary of Theorem 1.1:

Corollary 1.2. *Let K denote the Cantor set. There exists $C > 0$ such that for each $x \in K \setminus \mathbb{Q}$, there exist infinitely many $p/q \in \mathbb{Q} \setminus K$ satisfying*

$$(1.2) \quad \left| x - \frac{p}{q} \right| \leq \frac{C}{q^2}.$$

Remark. The hypotheses of Theorem 1.1 can be weakened. Recall that for $\varepsilon > 0$, a subset S of a metric space X is said to be ε -porous relative to X if for every ball $B(x, r) \subseteq X$, there exists a ball $B(y, \varepsilon r) \subseteq B(x, r)$ which is disjoint from S . Using this definition, Theorem 1.1 can be generalized as follows:

Theorem 1.3. *Fix $d \in \mathbb{N}$, and let $K \subseteq \mathbb{R}^d$ be a compact set whose intersection with every line $L \subseteq \mathbb{R}^d$ is ε -porous relative to L , for some $\varepsilon > 0$ independent of L . Then there exists $C > 0$ such that for all $\mathbf{x} \in K \setminus \mathbb{Q}^d$, there are infinitely many $\mathbf{p}/q \in \mathbb{Q}^d \setminus K$ satisfying (1.1).*

Theorem 1.3 is readily seen to be equivalent to Corollary 2.7 below. The deduction of Theorem 1.1 from Corollary 2.7 (or equivalently from Theorem 1.3) is given in Section 3.

We remark that the condition on K is satisfied whenever K is the support of an Ahlfors regular measure of dimension strictly less than 1.⁴ Moreover, if $d = 1$ the condition just reduces to K itself being porous, a fairly weak geometric condition (for example it is closed under quasiconformal maps).

1.2. The line segment hypothesis. A key hypothesis of Theorem 1.1 is that the set S does not contain a line segment. This hypothesis is not at all automatic; there exist examples of both manifolds and fractals which contain line segments. In the case of fractals, the Sierpinski triangle and the Sierpinski carpet are two examples of well-known fractals each of which contains a line segment. In the case of manifolds, there are numerous examples in \mathbb{R}^3 of so-called “ruled surfaces” which are in fact the union of lines.

In light of the above facts, one might ask whether the line segment hypothesis can be removed. However, it is necessary for the following simple reason:

²In this paper all iterated function systems are finite and consist of similarities.

³An analogue of Dirichlet's theorem for which the function $\mathbf{x} \mapsto C_{\mathbf{x}}$ is unbounded was already considered in [11, Theorem 8.1]. In the present case the situation is somewhat better, since the function $\mathbf{x} \mapsto C_{\mathbf{x}}$ is bounded on compact sets.

⁴A measure μ on a metric space X is said to be *Ahlfors regular of exponent δ* if there exists $C > 0$ such that for all $x \in \text{Supp}(\mu)$ and $0 < r \leq 1$, $(1/C)r^\delta \leq \mu(B(x, r)) \leq Cr^\delta$.

Observation 1.4. If S contains a rational line segment L , then the conclusion of Theorem 1.1 cannot hold.

Proof. Fix $\mathbf{x} \in L \setminus \mathbb{Q}^d \subseteq S \setminus \mathbb{Q}^d$. Then for all $\mathbf{p}/q \in \mathbb{Q}^d \setminus S \subseteq \mathbb{Q}^d \setminus L$,

$$\left\| \mathbf{x} - \frac{\mathbf{p}}{q} \right\| \geq d \left(\frac{\mathbf{p}}{q}, L \right) \gtrsim_L \frac{1}{q}.$$

For q sufficiently large, this contradicts (1.1). \square

In particular, the Sierpinski triangle and the Sierpinski carpet mentioned above each contain the interval $[0, 1]$ viewed as a subset of the x -axis, which is a rational line segment. Thus they cannot satisfy the conclusion of Theorem 1.1.

It is therefore a relevant question which manifolds and fractals contain a line segment, and which do not. In the case of manifolds, the condition can be translated into a differential condition, which we implicitly do in the proof of Claim 3.1 (cf. Remark 3.3). In the case of fractals, the condition is somewhat harder to check directly. On the other hand, many fractals are totally disconnected; no such fractal can contain a line segment. To give an example of how one can check that a fractal Λ does not contain a line segment in a case where Λ is not totally disconnected, we demonstrate the following:

Proposition 1.5. *The von Koch snowflake curve does not contain a line segment.*

It seems likely that the techniques used in the proof of Proposition 1.5 can be generalized, but it is not clear what the statement of the generalization should be.

Overview. Sections 2-3 are devoted to the proof of Theorem 1.1, with Section 2 containing preliminaries and Section 3 containing the body of the proof. Section 4 contains the proof of Proposition 1.5. In Section 5, we describe some results regarding extrinsic Diophantine approximation which are corollaries of known theorems, and give some open problems.

2. SKETCH OF A PROOF OF COROLLARY 1.2; PRELIMINARIES

Before presenting the proofs of Theorem 1.1, we sketch a short proof of Corollary 1.2 which uses the theory of continued fractions. This proof contains the basic idea of the more general proof, but it has the advantage of being more intuitive to someone familiar with the theory of continued fractions. After sketching the proof of Corollary 1.2, we begin the preliminaries for the proof of Theorem 1.1, using the sketch as motivation.

2.1. Sketch of a proof of Corollary 1.2. We recall some elements of the theory of continued fractions (e.g. [16]). As a matter of notation, given $a_1, \dots \in \mathbb{N}$ we let

$$[0; a_1, \dots, a_n] = \frac{1}{a_1 + \frac{1}{\ddots + \frac{1}{a_n}}}$$

and let $[0; a_1, \dots] = \lim_{n \rightarrow \infty} [0; a_1, \dots, a_n]$. Any $x \in (0, 1) \setminus \mathbb{Q}$ can be represented uniquely as $x = [0; a_1, \dots]$ for some sequence $(a_n)_1^\infty \in \mathbb{N}^\mathbb{N}$. The rationals $p_n/q_n = [0; a_1, \dots, a_n]$ are called the *convergents* of x . We have the following (see e.g. [16, (30) and Theorem 19]):

$$(2.1) \quad \left| x - \frac{p}{q} \right| < \frac{1}{2q^2} \Rightarrow \frac{p}{q} \text{ is a convergent of } x \Rightarrow \left| x - \frac{p}{q} \right| < \frac{1}{q^2}.$$

Now let K denote the Cantor set, and suppose $x \in K \setminus \mathbb{Q}$. If all but finitely many convergents of x lie in the Cantor set, then (2.1) shows that Corollary 1.2 cannot be true with $C = 1/2$; if infinitely many convergents of x lie outside of the Cantor set, then (2.1) shows that the conclusion of Corollary 1.2 holds with $C = 1$. However, the question of whether infinitely many convergents of x must lie outside the Cantor set is a difficult question, which we do not attempt to address here. Instead, we adopt the more modest approach of looking beyond the set of convergents; we are content to raise the value of C if necessary.

Sketch of the proof of Corollary 1.2. (For a rigorous proof of Corollary 1.2, see Section 3.) Let K denote the Cantor set, and suppose $x = [0; a_1, \dots] \in K \setminus \mathbb{Q}$. Fix $n \in \mathbb{N}$, and for each $b \in \mathbb{N}$ let

$$\frac{p_{n,b}}{q_{n,b}} = [0; a_1, \dots, a_{n-1}, b].$$

Then $p_{n,a_n}/q_{n,a_n} = p_n/q_n$, but otherwise $p_{n,b}/q_{n,b}$ is not a convergent of x . Now fix $N \in \mathbb{N}$, and consider the finite sequence $(p_{n,b}/q_{n,b})_{b=a_n}^{a_n+N}$. It can be shown that

- (i) $q_{n,b} \asymp_N q_n$ for $b = a_n, \dots, a_n + N$, and that
- (ii) the sequence $(p_{n,b}/q_{n,b})_{b=a_n}^{a_n+N}$ is roughly an arithmetic progression of increment $1/q_n^2$.

(The notion of a “roughly arithmetic progression” will be made precise in Definition 2.5.) In particular, for each $b = a_n, \dots, a_n + N$,

$$\left| x - \frac{p_{n,b}}{q_{n,b}} \right| \lesssim \frac{|a_n - b| + 1}{q_n^2},$$

and so

$$\left| x - \frac{p_{n,b}}{q_{n,b}} \right| \lesssim_N \frac{1}{q_{n,b}^2}.$$

Let C_N be the implied constant. If we assume for a contradiction that (1.2) holds for only finitely many p/q , then there exists n such that for all $b = a_n, \dots, a_n + N$, we have $p_{n,b}/q_{n,b} \in K$. It follows that K contains arbitrarily large roughly arithmetic progressions.

On the other hand, it is easily seen that K does not contain any arithmetic progression of length five. (If such a progression existed, then the largest such progression would need to contain points from both $[0, 1/3]$ and $[2/3, 1]$, so its increment would need to be at least $1/3$.) A similar argument shows that there exists N for which K does not contain any roughly arithmetic progressions of length N (cf. Proposition 2.6 below). This is a contradiction. \square

2.2. Good pairs of rational approximations. In higher dimensions, we cannot use the theory of continued fractions, but we will still produce a sequence of roughly arithmetic progressions which consist of good rational approximations to the desired point. To see how this generalization will work, note that we can write (see e.g. [16, Theorem 1])

$$\frac{p_{n,b}}{q_{n,b}} = \frac{p_{n-2} + bp_{n-1}}{q_{n-2} + bq_{n-1}}.$$

In particular, $((p_{n,b}, q_{n,b}))_{b \in \mathbb{N}}$ is a true arithmetic progression in \mathbb{Z}^2 , whose initial value (p_{n-2}, q_{n-2}) and increment (p_{n-1}, q_{n-1}) both represent good rational approximations to x . In higher dimensions, we will use the same principle, taking an arithmetic progression in \mathbb{Z}^{d+1} and projectivizing to get a roughly arithmetic progression in \mathbb{Q}^d .

The initial value and increment of our progression must both be good approximations, but they will not be chosen independently; they should be roughly “on the same order of magnitude”. We make this rigorous in the following lemma:

Lemma 2.1. *Fix $\mathbf{x} \in \mathbb{R}^d \setminus \mathbb{Q}^d$. Then for every $Q > 0$, there exists a pair $(\mathbf{r}_0, \mathbf{r}_\infty) \in (\mathbb{Z}^{d+1})^2$ such that*

- (i) $\mathbf{r}_0, \mathbf{r}_\infty$ are linearly independent;
- (ii) If we write $\mathbf{r}_i = (\mathbf{p}_i, q_i)$ ($i = 0, \infty$), then $0 \leq q_\infty \leq q_0$ and

$$(2.2) \quad \|q_i \mathbf{x} - \mathbf{p}_i\| \leq \frac{1}{q_0^{1/d}};$$

and such that $q_0 \geq Q$.

We will call a pair $(\mathbf{r}_0, \mathbf{r}_\infty)$ satisfying (i) and (ii) a *good pair* for \mathbf{x} .

Proof of Lemma 2.1. Interpret \mathbf{x} as a column vector and let

$$T_{\mathbf{x}} := \begin{bmatrix} I_d & -\mathbf{x} \\ 0 & 1 \end{bmatrix}$$

$$g_t := \begin{bmatrix} e^{t/d} I_d & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

Here I_d denotes the $d \times d$ identity matrix.

Claim 2.2. *There exists a sequence $t_k \xrightarrow[k]{} \infty$ such that for each $k \in \mathbb{N}$,*

$$\lambda_2(g_{t_k} \circ T_{\mathbf{x}}(\mathbb{Z}^{d+1})) \leq 1,$$

where λ_2 is the second successive Minkowski minimum⁵ (with respect to the max norm).

Proof. By contradiction, suppose that there exists $t_0 > 0$ such that for all $t > t_0$ we have $\lambda_2(g_t \circ T_{\mathbf{x}}(\mathbb{Z}^{d+1})) > 1$, and let $U = (t_0, \infty)$.

Let $\mathbb{Z}_{\text{pr}}^{d+1}$ denote the set of primitive vectors of \mathbb{Z}^{d+1} . For each $\mathbf{r} \in \mathbb{Z}_{\text{pr}}^{d+1}$ let

$$U_{\mathbf{r}} = \{t \in U : \|g_t \circ T_{\mathbf{x}}(\mathbf{r})\| < 1\}.$$

We claim that the collection of sets $(U_{\mathbf{r}})_{\mathbf{r} \in \mathbb{Z}_{\text{pr}}^{d+1}}$ is a disjoint open cover of U . Indeed, if $t \in U_{\mathbf{r}_1} \cap U_{\mathbf{r}_2}$ for some $\mathbf{r}_1, \mathbf{r}_2 \in \mathbb{Z}_{\text{pr}}^{d+1}$ for which $U_{\mathbf{r}_1} \neq U_{\mathbf{r}_2}$, then we would have $\lambda_2(g_t \circ T_{\mathbf{x}}(\mathbb{Z}^{d+1})) < 1$, contradicting our hypothesis. On the other hand, for any $t \in U$, we have by Minkowski's second theorem

$$\lambda_1(g_t \circ T_{\mathbf{x}}(\mathbb{Z}^{d+1})) \leq \frac{1}{\prod_{i=2}^{d+1} \lambda_i(g_t \circ T_{\mathbf{x}}(\mathbb{Z}^{d+1}))} \leq \frac{1}{\lambda_2(g_t \circ T_{\mathbf{x}}(\mathbb{Z}^{d+1}))^d} < 1,$$

and so there exists $\mathbf{r} \in \mathbb{Z}_{\text{pr}}^{d+1}$ such that $\|g_t \circ T_{\mathbf{x}}(\mathbf{r})\| < 1$, i.e. $t \in U_{\mathbf{r}}$.

Since U is a connected set, it follows that $U = U_{\mathbf{r}}$ for some $\mathbf{r} \in \mathbb{Z}_{\text{pr}}^{d+1}$. Then $\|g_t \circ T_{\mathbf{x}}(\mathbf{r})\|$ is bounded as t tends to infinity. Thus $T_{\mathbf{x}}(\mathbf{r}) \in \mathbb{R}\mathbf{e}_{d+1}$, i.e. $\mathbf{x} = \mathbf{p}/q$ where $\mathbf{r} = (\mathbf{p}, q)$, contradicting that $\mathbf{x} \notin \mathbb{Q}^d$. \triangleleft

Now fix $k \in \mathbb{N}$ and let $t = t_k$. Since $\lambda_2(g_t \circ T_{\mathbf{x}}(\mathbb{Z}^{d+1})) \leq 1$, there exist $\mathbf{r}_0, \mathbf{r}_{\infty} \in \mathbb{Z}^{d+1}$ linearly independent such that

$$\|g_t \circ T_{\mathbf{x}}(\mathbf{r}_i)\| \leq 1, \quad i = 0, \infty.$$

Writing this out in terms of coordinates yields

$$\|e^{t/d}(\mathbf{p}_i - q_i \mathbf{x})\| \leq 1 \text{ and } |e^{-t} q_i| \leq 1,$$

or equivalently

$$(2.3) \quad \|q_i \mathbf{x} - \mathbf{p}_i\| \leq e^{-t/d} \text{ and } |q_i| \leq e^t.$$

By replacing \mathbf{r}_i by $-\mathbf{r}_i$ if necessary, we may assume $q_i \geq 0$ for $i = 0, \infty$; by swapping \mathbf{r}_0 and \mathbf{r}_{∞} if necessary we may assume $q_{\infty} \leq q_0$. After these reductions, (2.3) implies (2.2), which demonstrates that the pair $(\mathbf{r}_0, \mathbf{r}_{\infty})$ is good for \mathbf{x} . Finally, we observe that as $k \rightarrow \infty$ we have $q_0 \rightarrow \infty$, because otherwise (2.3) would imply that $\mathbf{x} \in \mathbb{Q}^d$. Thus for any given $Q > 0$, we can construct a good pair for which $q_0 \geq Q$. \square

Suppose that $(\mathbf{r}_0, \mathbf{r}_{\infty})$ is a good pair for \mathbf{x} . We now create an arithmetic progression in \mathbb{Z}^{d+1} using \mathbf{r}_0 as the initial value and \mathbf{r}_{∞} as the increment. For each $i \in \mathbb{Z}$ let

$$(2.4) \quad \mathbf{r}_i = \mathbf{r}_0 + i\mathbf{r}_{\infty}$$

and write

$$\mathbf{r}_i = (\mathbf{p}_i, q_i).$$

Then $\{\mathbf{r}_i : i \in \mathbb{Z}\}$ is an arithmetic progression in \mathbb{Z}^{d+1} .

Our key claim is that *every* rational \mathbf{p}_i/q_i represents a good approximation to \mathbf{x} in the sense of (1.1), with C depending only on i and not on \mathbf{x} .

⁵See e.g. [18, §IV.1] for an exposition of Minkowski's theory of successive minima.

Claim 2.3. For $i \in \mathbb{N}$,

$$(2.5) \quad \left\| \mathbf{x} - \frac{\mathbf{p}_i}{q_i} \right\| \leq \left(\frac{1+i}{q_i} \right)^{1+1/d}.$$

Proof.

$$\begin{aligned} \|q_i \mathbf{x} - \mathbf{p}_i\| &= \|(q_0 \mathbf{x} - \mathbf{p}_0) + i(q_\infty \mathbf{x} - \mathbf{p}_\infty)\| \\ &\leq \frac{1+i}{q_0^{1/d}} && \text{(by (2.2))} \\ q_i &= q_0 + i q_\infty \leq (1+i)q_0 \end{aligned}$$

and rearranging gives the desired result. \square

We remark that for $N \in \mathbb{N}$ fixed, if we let $C_N = (1+N)^{1+1/d}$, then (2.5) implies (1.1) for $i = 0, \dots, N$.

2.3. Roughly arithmetic progressions. We would now like to make rigorous in what sense the sequence $(\mathbf{p}_i/q_i)_0^\infty$ described above is a roughly arithmetic progression. We begin with the following observation:

Observation 2.4. For each $i \in \mathbb{Z}$, \mathbf{p}_i/q_i is on the line spanned by \mathbf{p}_0/q_0 and $\mathbf{p}_\infty/q_\infty$.

Proof. The line spanned by \mathbf{p}_0/q_0 and $\mathbf{p}_\infty/q_\infty$ is the projectivization of the two-dimensional subspace of \mathbb{R}^{d+1} spanned by \mathbf{r}_0 and \mathbf{r}_∞ . \square

Thus, it will not be too restrictive for us to require roughly arithmetic progressions to be subsets of lines.

Let L be a line in \mathbb{R}^d , and let L_0 be its linear part, i.e. $L_0 = L - \mathbf{x}$ where $\mathbf{x} \in L$ is any point. One way of defining arithmetic progressions on L is to say that a sequence $(\mathbf{x}_i)_0^N$ is an arithmetic progression if there exists a vector $\mathbf{v} \in L_0$ (the *increment*) such that for all $0 \leq i < j \leq N$,

$$(2.6) \quad \mathbf{x}_j - \mathbf{x}_i = (j-i)\mathbf{v}.$$

To define a roughly arithmetic progression, we will relax the condition (2.6). Specifically, we have the following:

Definition 2.5. Fix $C > 0$. A sequence $(\mathbf{x}_i)_0^N$ in L is a C -roughly arithmetic progression if there exists $\mathbf{v} \in L_0 \setminus \{\mathbf{0}\}$ such that for all $0 \leq i < j \leq N$,

$$(2.7) \quad \frac{1}{C}(j-i) \leq \frac{\mathbf{x}_j - \mathbf{x}_i}{\mathbf{v}} \leq C(j-i).$$

Here the expression $\frac{\mathbf{x}_j - \mathbf{x}_i}{\mathbf{v}}$ denotes the unique value $c \in \mathbb{R}$ for which $\mathbf{x}_j - \mathbf{x}_i = c\mathbf{v}$.

In fact, the sequence $(\mathbf{p}_i/q_i)_0^\infty$ is not a roughly arithmetic progression in this sense, since $\|\mathbf{p}_j/q_j - \mathbf{p}_i/q_i\| \rightarrow 0$ as $i, j \rightarrow \infty$. However, we will now prove that sufficiently long subsequences of the sequence $(\mathbf{p}_i/q_i)_0^\infty$ are roughly arithmetic.

Proposition 2.6. Fix $\mathbf{p}_0/q_0, \mathbf{p}_\infty/q_\infty \in \mathbb{Q}^d$, and for each $i \in \mathbb{N}$ let $\mathbf{p}_i = \mathbf{p}_0 + i\mathbf{p}_\infty$ and $q_i = q_0 + iq_\infty$. Then for each $N \in \mathbb{N}$, $(\mathbf{p}_i/q_i)_{N^{2N}}^{2N}$ is a 2-roughly arithmetic progression.

Proof. We observed above (Observation 2.4) that the sequence $(\mathbf{p}_i/q_i)_{N^{2N}}^{2N}$ is collinear. Fix $N \leq i < j \leq 2N$. Then

$$\begin{aligned} \frac{\mathbf{p}_j}{q_j} - \frac{\mathbf{p}_i}{q_i} &= \frac{1}{q_i q_j} [q_i \mathbf{p}_j - q_j \mathbf{p}_i] \\ &= \frac{1}{q_i q_j} [(q_0 + iq_\infty)(\mathbf{p}_0 + j\mathbf{p}_\infty) - (q_0 + jq_\infty)(\mathbf{p}_0 + i\mathbf{p}_\infty)] \\ &= \frac{1}{q_i q_j} (j-i) [q_0 \mathbf{p}_\infty - q_\infty \mathbf{p}_0]. \end{aligned}$$

So let

$$\mathbf{v} = \frac{q_0 \mathbf{p}_\infty - q_\infty \mathbf{p}_0}{q_N q_{2N}},$$

so that

$$\frac{\frac{\mathbf{p}_j}{q_j} - \frac{\mathbf{p}_i}{q_i}}{\mathbf{v}} = \frac{q_N q_{2N}}{q_i q_j}.$$

The proposition follows on noting that

$$q_{2N} = q_0 + 2Nq_\infty \leq 2q_0 + 2Nq_\infty = 2q_N.$$

□

Proposition 2.6 allows us to prove a preliminary version of Theorem 1.1:

Corollary 2.7. *Fix $d \in \mathbb{N}$ and a set $S \subseteq \mathbb{R}^d$. Suppose that for some N , S contains no 2-roughly arithmetic progression of length N . Then there exists $C > 0$ such that for all $\mathbf{x} \in S \setminus \mathbb{Q}^d$, there exist infinitely many $\mathbf{p}/q \in \mathbb{Q}^d \setminus S$ satisfying*

$$(1.1) \quad \left\| \mathbf{x} - \frac{\mathbf{p}}{q} \right\| \leq \frac{C}{q^{1+1/d}}.$$

In other words, if S contains no 2-roughly arithmetic progression of length N , then S satisfies the conclusion of Theorem 1.1 with C independent of \mathbf{x} .

Proof. Let $C = C_N = (1 + N)^{1+1/d}$, and fix $\mathbf{x} \in S \setminus \mathbb{Q}^d$.

Fix a pair $(\mathbf{r}_0, \mathbf{r}_\infty)$ which is good for \mathbf{x} , and for each $i \in \mathbb{N}$, let $\mathbf{r}_i = (\mathbf{p}_i, q_i)$ be defined by (2.4). By Proposition 2.4, the sequence $(\mathbf{p}_i/q_i)_{i=0}^{2N}$ is a 2-roughly arithmetic progression, so by hypothesis, this sequence contains a point which is not in S , say $\mathbf{p}_i/q_i \notin S$. By Claim 2.3, (1.1) is satisfied for \mathbf{p}_i/q_i . To summarize, for each good pair $(\mathbf{r}_0, \mathbf{r}_\infty)$, there is a rational $\mathbf{p}_i/q_i \in \mathbb{Q}^d \setminus S$ satisfying (1.1) with $q_i \geq q_0$.

By Lemma 2.1, for each $Q > 0$ there is a good pair satisfying $q_0 \geq Q$. Thus by the above argument, there exists $\mathbf{p}/q \in \mathbb{Q}^d \setminus S$ satisfying (1.1) such that $q \geq Q$. Thus there are infinitely many rationals $\mathbf{p}/q \in \mathbb{Q}^d \setminus S$ satisfying (1.1). □

Corollary 2.7 says that in order to prove the extrinsic analogue for points in a fractal or manifold, it is enough to demonstrate a uniform bound on the length of a 2-roughly arithmetic progression contained in that set. Intuitively, the reason for this should be that S contains no line segment by assumption. (If S did contain a line segment, then it would automatically contain arbitrarily long arithmetic progressions, which would in particular be C -roughly arithmetic for every $C \geq 1$.) So it will be useful to know that in certain cases, the limit of roughly arithmetic progressions is a line segment. To make this rigorous, we recall the definition of the *Hausdorff metric* on the space of compact subsets of a metric space X .

Definition 2.8. Let $\mathcal{K}^*(X)$ denote the set of nonempty compact subsets of X . The *Hausdorff distance* between two sets $K_1, K_2 \in \mathcal{K}^*(X)$ is the number

$$d_H(K_1, K_2) := \max \left\{ \max_{x \in K_1} d(x, K_2), \max_{x \in K_2} d(x, K_1) \right\}.$$

For background on the Hausdorff metric, see [15, §4.F]. The topology induced on $\mathcal{K}^*(X)$ by the Hausdorff metric is called the *Vietoris topology* (cf. [15, Exercise 4.21]).

We may now prove the following lemma:

Lemma 2.9. *Fix $C > 0$. For each N , suppose that $K_N \subseteq \mathbb{R}$ is a C -roughly arithmetic progression of length $(N + 1)$, whose left and right endpoints are equal to 0 and 1, respectively. Then*

$$K_N \xrightarrow{N} [0, 1]$$

in the Vietoris topology.

Proof. Write $K_N = (x_{N,i})_{i=0}^N$. Then (2.7) reads:

$$\frac{j-i}{C} \leq \frac{x_{N,j} - x_{N,i}}{v_N} \leq C(j-i).$$

Plugging in $i = 0, j = N$ shows that $0 < v_N \leq C/N$. Then, plugging in $j = i+1$ shows that $x_{N,i+1} - x_{N,i} \leq C^2/N$ for all $i = 0, \dots, N-1$. It follows that $[0, 1] \setminus K_N$ cannot contain any interval of length greater than C^2/N . In particular,

$$d(x, K_N) \leq C^2/(2N) \quad \forall x \in [0, 1].$$

Since $K_N \subseteq [0, 1]$, this implies that $d_H(K_N, [0, 1]) \leq C^2/(2N)$. Since $C^2/(2N) \rightarrow 0$ as $N \rightarrow \infty$, this completes the proof. \square

2.4. Iterated function systems. We now recall the notion of an iterated function system (IFS); for a detailed exposition see [10, §9]. We will only consider the case of a finite IFS generated by similarities.

Definition 2.10. Fix $d \in \mathbb{N}$, and let E be a finite set. An *iterated function system (IFS)* on \mathbb{R}^d is a collection $(u_a)_{a \in E}$ of contracting similarities $u_a : \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfying the *open set condition*: there exists an open set $W \subseteq \mathbb{R}^d$ such that the collection $(u_a(W))_{a \in E}$ is a disjoint collection of subsets of W (see [14] for a thorough discussion). The *limit set* of the IFS is the image of the *coding map* $\pi : E^{\mathbb{N}} \rightarrow \mathbb{R}^d$ defined by

$$\pi(\omega) = \lim_{n \rightarrow \infty} u_{\omega_1} \circ \dots \circ u_{\omega_n}(0).$$

Remark 2.11. By intersecting with a ball centered at $\mathbf{0}$ of sufficiently large radius, we may without loss of generality assume that the open set W is bounded.

Let us now introduce some notation. Let $E^* = \bigcup_{n \geq 0} E^n$. For $\omega \in E^*$, let $|\omega|$ denote the length of ω , and let

$$u_\omega = u_{\omega_1} \circ \dots \circ u_{\omega_{|\omega|}},$$

with the convention that u_\emptyset is the identity map.

Although we believe the following lemma is well-known to experts, we include its proof for completeness.

Lemma 2.12. *Let $(u_a)_{a \in E}$ be an IFS, and let Λ denote the limit set of $(u_a)_{a \in E}$. There exists a constant $M \in \mathbb{N}$ such that for every set $S \subseteq \Lambda$, there exists a collection $A \subseteq E^*$ of cardinality at most M with the following properties:*

- (i) $S \subseteq \bigcup_{\omega \in A} u_\omega(\Lambda)$.
- (ii) For all $\omega \in A$,

$$(2.8) \quad \|u'_\omega\| \leq \text{diam}(S),$$

where $\|u'_\omega\|$ denotes the contraction ratio of the similarity u_ω .

Proof. Let \tilde{A} be the set of all words $\omega \in E^*$ which satisfy (2.8) but for which no proper initial segment satisfies (2.8), and let $A = \{\omega \in \tilde{A} : u_\omega(\Lambda) \cap S \neq \emptyset\}$. Then (i) and (ii) are satisfied. To complete the proof, we must show that $\#(A)$ is bounded independent of S .

If $\emptyset \in A$, then $\#(A) = 1$. Thus we may assume $\emptyset \notin A$. Fix $\omega \in A$. The minimality of ω implies that

$$\|u'_{\omega \upharpoonright [1, |\omega|-1]}\| > \text{diam}(S),$$

where \upharpoonright denotes restriction. In particular, letting $\gamma = \min_{a \in E} \|u'_a\| > 0$, we have

$$\|u'_\omega\| = \|u'_{\omega \upharpoonright [1, |\omega|]}\| \cdot \|u'_{\omega \upharpoonright [1, |\omega|-1]}\| > \gamma \text{diam}(S).$$

Moreover, if $\mathbf{x} \in S$ then

$$u_\omega(W) \subseteq B(\mathbf{x}, \text{diam}(S) + \text{diam}(u_\omega(W))) \subseteq B(\mathbf{x}, \text{diam}(S)(1 + \text{diam}(W))).$$

Since no word in A is an initial segment of another word in A , the open set condition implies that the collection $(u_\omega(W))_{\omega \in A}$ is disjoint. Letting λ denote Lebesgue measure, we have

$$\begin{aligned} \text{diam}(S)^d &\asymp \lambda(B(\mathbf{x}, \text{diam}(S)(1 + \text{diam}(W)))) \geq \sum_{\omega \in A} \lambda(u_\omega(W)) \\ &= \sum_{\omega \in A} \|u'_\omega\|^d \lambda(W) \asymp \sum_{\omega \in A} \text{diam}(S)^d. \end{aligned}$$

Dividing both sides by $\text{diam}(S)^d$, we see that $\#(A)$ is bounded from above independent of S . \square

3. PROOF OF THEOREM 1.1 (EXTRINSIC ANALOGUE OF DIRICHLET'S THEOREM)

Proof of Theorem 1.1, case (1). Suppose that S is the limit set of the IFS $(u_a)_{a \in E}$, and write $\Lambda = S$. By Corollary 2.7, to complete the proof it suffices to show that there exists N such that Λ contains no 2-roughly arithmetic progression of length N .

By contradiction, suppose that Λ contains arbitrarily long 2-roughly arithmetic progressions. For each N , let $P_N = (\mathbf{x}_i)_0^N$ be a 2-roughly arithmetic progression of length $(N+1)$, and let $\gamma_N : \mathbb{R} \rightarrow \mathbb{R}^d$ be an affine transformation such that $\gamma_N(0) = \mathbf{x}_0$ and $\gamma_N(1) = \mathbf{x}_N$. Then $K_N := \gamma_N^{-1}(P_N)$ is also a 2-roughly arithmetic progression; moreover, the left and right endpoints of K_N are 0 and 1, respectively.

Let $M \in \mathbb{N}$ be as in Lemma 2.12. Then by Lemma 2.12, for each $N \in \mathbb{N}$ there is a collection $A_N \subseteq E^*$ of cardinality at most M such that

$$(3.1) \quad P_N \subseteq \bigcup_{\omega \in A_N} u_\omega(\Lambda)$$

and

$$(3.2) \quad \|u'_\omega\| \leq \text{diam}(P_N) = \|\gamma'_N\| \quad \forall \omega \in A_N.$$

Enumerate the elements of A_N by $\omega^{(N,1)}, \dots, \omega^{(N,M_N)}$ with $M_N \leq M$. For each $j = 1, \dots, M$ let

$$K_{N,j} = \{x \in K_N : \gamma_N(x) \in u_{\omega^{(N,j)}}(\Lambda)\}$$

if $j \leq M_N$, and $K_{N,j} = \emptyset$ otherwise. By (3.1),

$$K_N = \bigcup_{j=1}^M K_{N,j}.$$

By the compactness of $\mathcal{K}^*([0,1])$ under the Vietoris topology [15, Theorem 4.26], there exists an increasing sequence $(N_k)_1^\infty$ such that for each $j = 1, \dots, M$, the sequence $(K_{N_k,j})_{k=1}^\infty$ converges to a set $K_{\infty,j} \in \mathcal{K}^*([0,1])$. Since the finite union operation is continuous in the Vietoris topology [15, Exercise 4.29(iv)], by Lemma 2.9 we have

$$\bigcup_{j=1}^M K_{\infty,j} = \lim_{N \rightarrow \infty} K_N = [0,1].$$

Now by elementary topology, the union of nowhere dense sets is nowhere dense, and so there exists $j = 1, \dots, M$ such that the set $K_{\infty,j}$ contains a nontrivial interval $[a,b] \subseteq [0,1]$. For each $N \in \mathbb{N}$, define $h_N : [0,1] \rightarrow \mathbb{R}^d$ by

$$h_N = u_{\omega^{(N,j)}}^{-1} \circ \gamma_N.$$

By the definition of $K_{N,j}$, we have $h_N(K_{N,j}) \subseteq \Lambda$. On the other hand, by (3.2) we have

$$\|h'_N\| \geq 1.$$

Since $h_N(0)$ and $h_N(1)$ are in the bounded set Λ , $\|h'_N\|$ is bounded independent of N . Let $(N_k)_1^\infty$ be an increasing sequence which is a subsequence of the previously chosen sequence and for which the sequence of affine functions h_{N_k} converges locally uniformly to a non-constant affine function $h : [0,1] \rightarrow \mathbb{R}^d$. The map $(h, K) \mapsto h(K)$ is continuous from (locally uniform topology \times Vietoris topology) to the Vietoris topology [19, (16.11)]; thus

$$h_{N_k}(K_{N_k,j}) \xrightarrow[k]{} h(K_{\infty,j}).$$

But $h_{N_k}(K_{N_k,j}) \subseteq \Lambda$ by construction, so $h(K_{\infty,j}) \subseteq \Lambda$ by [15, Exercise 4.29(ii)]. Since $K_{\infty,j} \supseteq [a, b]$, we have

$$\Lambda \supseteq h([a, b]),$$

i.e. Λ contains a line segment, contradicting our hypothesis. \square

Proof of Theorem 1.1, case (2). Write $M = S$. The first step of the proof is to show that since M does not contain a line segment, the cardinality of its intersection with any short enough line segment is bounded from above. Rigorously:

Claim 3.1. *For each $\mathbf{x} \in M$, there exist a neighborhood U of \mathbf{x} and an integer $N \in \mathbb{N}$ such that for every line L ,*

$$\#(U \cap L) \leq N.$$

Proof. By the implicit function theorem, if $U \subseteq \mathbb{R}^d$ is a sufficiently small neighborhood of \mathbf{x} , then there exist real-analytic functions $f_1, \dots, f_s : U \rightarrow \mathbb{R}$ such that $M \cap U = \bigcap_{i=1}^s f_i^{-1}(0)$, where $s = d - \dim(M)$. By contradiction, for each $N \in \mathbb{N}$ large enough so that $B(\mathbf{x}, 1/N) \subseteq U$, choose a line L_N so that

$$\#(M \cap B(\mathbf{x}, 1/N) \cap L_N) > N.$$

Parameterize L_N by an affine transformation $\gamma_N : \mathbb{R} \rightarrow \mathbb{R}^d$ satisfying $\|\gamma'_N\| = 1$. Since the unit sphere S^{d-1} is compact, we may choose a sequence $(N_k)_1^\infty$ and a vector $\mathbf{v} \in S^{d-1}$ so that $\gamma'_{N_k} \xrightarrow{k} \mathbf{v}$. Let $(a_N, b_N) = \gamma_N^{-1}(B(\mathbf{x}, 1/N))$.

Fix $i = 1, \dots, s$ and $N \in \mathbb{N}$. Then $f_i \circ \gamma_N$ has at least N zeros on (a_N, b_N) , since each point in $M \cap B(\mathbf{x}, 1/N) \cap L_N$ corresponds to a joint zero of f_1, \dots, f_s on (a_N, b_N) .

Claim 3.2. *For each $j \leq N$, $(f_i \circ \gamma_n)^{(j)}$ has at least $(N - j)$ zeros on (a_N, b_N) .*

Proof. Suppose the claim is true for $j < N$, and let $a_N < c_1 < \dots < c_{N-j} < b_N$ be zeros of $(f_i \circ \gamma_n)^{(j)}$. By the mean value theorem, for each $k = 1, \dots, N - (j + 1)$ there exists $c'_k \in (c_k, c_{k+1})$ which is a zero of $(f_i \circ \gamma_n)^{(j+1)}$. This completes the inductive step. \triangleleft

Now fix $j < N$, and let $c_{N,j} \in (a_N, b_N)$ be a zero of $(f_i \circ \gamma_N)^{(j)}$. We observe that by the chain rule,

$$0 = (f_i \circ \gamma_N)^{(j)}(c_{N,j}) = f_i^{(j)} \circ \gamma_N(c_{N,j})[\gamma'_N, \dots, \gamma'_N].$$

Here there are j copies of γ'_N . Note that we have used the fact that γ_N is affine to eliminate all terms involving a second order or higher derivative of γ_N , and to interpret γ'_N as a vector rather than as a function whose output is a vector. Since $\gamma_N(c_{N,j}) \in B(\mathbf{x}, 1/N)$, we have by Taylor's theorem

$$(3.3) \quad |f_i^{(j)}(\mathbf{x})[\gamma'_N, \dots, \gamma'_N]| \leq (1/N)^{j+1} \sup_{\mathbf{y} \in B(\mathbf{x}, 1/N)} \|f_i^{(j+1)}(\mathbf{y})\|.$$

On the other hand, since $\gamma'_{N_k} \xrightarrow{k} \mathbf{v}$ we have

$$f_i^{(j)}(\mathbf{x})[\gamma'_{N_k}, \dots, \gamma'_{N_k}] \xrightarrow{k} f_i^{(j)}(\mathbf{x})[\mathbf{v}, \dots, \mathbf{v}],$$

which together with (3.3) implies that

$$(3.4) \quad f_i^{(j)}(\mathbf{x})[\mathbf{v}, \dots, \mathbf{v}] = 0 \quad \forall i = 1, \dots, s \quad \forall j \in \mathbb{N}.$$

Since f_1, \dots, f_s are real-analytic, (3.4) implies that

$$f_i(\mathbf{x} + t\mathbf{v}) = 0 \quad \forall i = 1, \dots, s \quad \forall t \in \mathbb{R} \text{ sufficiently small.}$$

Reinterpreting this statement in terms of the manifold M , we see that for sufficiently small $\varepsilon > 0$, the line segment $\mathbf{x} + [-\varepsilon, \varepsilon]\mathbf{v}$ is contained in M . This contradicts our hypothesis. \triangleleft

Remark 3.3. In the above proof, and for the remainder of the proof of Theorem 1.1, the only step where we need M to be real-analytic is the step where we use (3.4) to deduce the existence of a line segment contained in M . If M is assumed to be \mathcal{C}^∞ and if we assume that (3.4) does not hold for any pair (\mathbf{x}, \mathbf{v}) , then the conclusion of Theorem 1.1 holds.

We now claim that for any compact set $K \subseteq M$, the conclusion of Theorem 1.1 holds, with the constant $C_{\mathbf{x}}$ depending only on K . Indeed, fix such a K , let $V \subseteq M$ be a neighborhood of K which is relatively compact in M , and let $\tilde{K} = \bar{V}$. We use the compactness of \tilde{K} to change the local principle of Claim 3.1 to a global one:

Claim 3.4. *There exists $N \in \mathbb{N}$ such that for every line L ,*

$$\#(\tilde{K} \cap L) < N.$$

Proof. For each $\mathbf{x} \in M$, let $U_{\mathbf{x}}$ and $N_{\mathbf{x}}$ be as in Claim 3.1. Then $(U_{\mathbf{x}})_{\mathbf{x} \in \tilde{K}}$ is a cover of \tilde{K} ; let $(U_{\mathbf{x}_i})_{i=1}^k$ be a finite subcover. The corollary holds with $N = \sum_{i=1}^k N_{\mathbf{x}_i} + 1$. \triangleleft

Thus \tilde{K} contains no collinear N -tuples of distinct points, and in particular, \tilde{K} contains no 2-roughly arithmetic progressions of length N . So by Corollary 2.7, there exists $C > 0$ such that for all $\mathbf{x} \in K \setminus \mathbb{Q}^d$, there exist infinitely many $\mathbf{p}/q \in \mathbb{Q}^d \setminus \tilde{K}$ satisfying (1.1). Since K is contained in the interior of \tilde{K} relative to M , only finitely many of these rational points can satisfy $\mathbf{p}/q \in M \setminus \tilde{K}$. Thus there are infinitely many $\mathbf{p}/q \in \mathbb{Q}^d \setminus M$ satisfying (1.1). \square

4. PROOF OF PROPOSITION 1.5 (THE VON KOCH CURVE DOES NOT CONTAIN A LINE SEGMENT)

Recall (cf. [14, §3.3(2)]) that the von Koch snowflake curve is the limit set of the IFS on $\mathbb{R}^2 \equiv \mathbb{C}$ generated by the similarities

$$\begin{aligned} u_1(z) &= \frac{1}{3}z \\ u_2(z) &= \frac{1}{3}e^{\pi i/3}z + \frac{1}{3} \\ u_3(z) &= \frac{1}{3}e^{-\pi i/3}z + \frac{1}{3} + \frac{1}{3}e^{2\pi i/3} \\ u_4(z) &= \frac{1}{3}z + \frac{2}{3}. \end{aligned}$$

This IFS satisfies the open set condition with respect to the equilateral triangle W whose vertices are 0, 1, and $e^{\pi i/3}$ (cf. Figure 1). Denote the von Koch curve by Λ .

Convention. In this proof, line segments are not considered to contain their endpoints.

By contradiction, suppose that the von Koch curve contains a line segment $L \subseteq \Lambda$. Without loss of generality, suppose that

- (i) The number of endpoints of L contained in $\bigcup_a \partial(u_a(W))$ is maximal among line segments contained in Λ .
- (ii) The length of L is maximal given (i).

We observe that L cannot be contained in $u_a(W)$ for any $a \in E$; otherwise $u_a^{-1}(L)$ would be a line segment longer than L but also satisfying (i). Since L is connected, it follows that $L \setminus \bigcup_a u_a(W) \neq \emptyset$. Fix $x \in L \setminus \bigcup_a u_a(W) \subseteq L \cap \bigcup_a \partial(u_a(W))$. Then $L \setminus \{x\}$ is the union of two line segments L^1 and L^2 . Let n denote the number of endpoints of L contained in $\bigcup_a \partial(u_a(W))$; we claim that $n = 2$. Indeed, if not, then either L^1 or L^2 has $(n+1)$ endpoints contained in $\bigcup_a \partial(u_a(W))$. (If $n = 0$, both L^1 and L^2 have this property; if $n = 1$, only one of them does.)

To summarize: both endpoints x_1, x_2 of the line segment L described by conditions (i) and (ii) are contained in $\bigcup_a \partial(u_a(W))$; moreover, L is not contained in $u_a(W)$ for any a . Let L_1, \dots, L_9 be as in Figure 1, so that $\bigcup_a \partial(u_a(W)) = \bigcup_1^9 L_i$. We now consider separately:

Case 1: x_1, x_2 are contained in the same line segment L_i for some i . In this case, we observe that since the intersection of Λ with the x -axis is precisely the Cantor set, the intersection of Λ with these line segments will be the union of finitely many images of the Cantor set under similarities. Since such a union cannot contain a line segment, this is a contradiction.

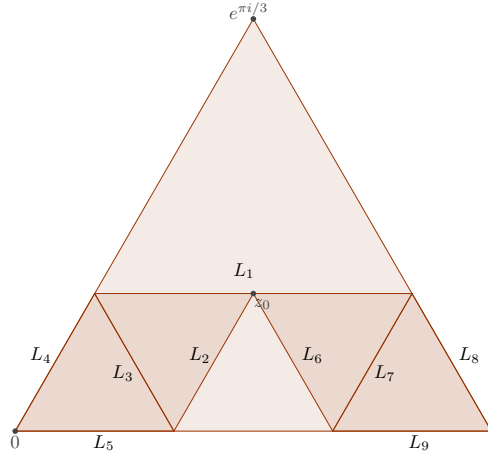


FIGURE 1. The open set and its first-level iterates for the von Koch snowflake curve.

- Case 2: x_1, x_2 lie on opposite sides of the line $L_{10} = \{\operatorname{Re}[z] = 1/2\}$. In this case, we observe that $\Lambda \cap L_{10}$ is a singleton $\{z_0\}$, where $z_0 = \frac{1}{3} + \frac{1}{3}e^{2\pi i/3}$ as in Figure 1. Since L is connected, we must have $z_0 \in L$. But since no point in Λ has imaginary part greater than the imaginary part of z_0 , the line L must be horizontal. Thus $L \subseteq L_1$, and we are reduced to the first case.
- Case 3: x_1, x_2 are contained in different line segments, and lie on the same side of L_{10} . In this case, without loss of generality we may assume that x_1, x_2 lie on the left hand side of L_{10} . Now if $\{x_1, x_2\} \subseteq L_1 \cup L_2 \cup L_3$ or $\{x_1, x_2\} \subseteq L_3 \cup L_4 \cup L_5$, then we would have $L \subseteq u_1(W)$ or $L \subseteq u_2(W)$, respectively. Either is a contradiction, so $\{x_1, x_2\} \not\subseteq L_1 \cup L_2 \cup L_3$ and $\{x_1, x_2\} \not\subseteq L_3 \cup L_4 \cup L_5$. It follows (after possibly swapping x_1 and x_2) that $x_1 \in L_1 \cap L_2 \setminus L_3$ and $x_2 \in L_4 \cup L_5 \setminus L_3$. In particular, we see that L can be written as a union $L = L^1 \cup \{y\} \cup L^2$, where $L^a \subseteq u_a(W)$ and $y \in L_3$. Now for one of $a = 1, 2$, the length of L^a is at least half of the length of L . But then $u_a^{-1}(L^a)$ is longer than L , contradicting (ii).

5. METRICAL EXTRINSIC APPROXIMATION

Theorem 1.1 gives an analogue of Dirichlet's theorem in the setting of extrinsic approximation. It is reasonable to ask whether analogues of the other classical theorems of Diophantine approximation, namely the Jarník–Schmidt, Khinchin, and Jarník–Besicovitch theorems (see e.g. [18, Theorem III.2A] and [7, Theorems 1.10 and 5.2]), also hold. In the case of the Jarník–Schmidt theorem, an extrinsic version can be deduced immediately from the ambient version, and in the case of Khinchin's theorem, an extrinsic version can be deduced from the ambient version together with a statement regarding intrinsic approximation which was proven in [11]. Finally, the Jarník–Besicovitch theorem is more subtle, and does not admit an extrinsic analogue with the same level of generality. We comment on this phenomenon in §5.3 below.

5.1. An analogue of the Jarník–Schmidt theorem. The analogue of the Jarník–Schmidt theorem for ambient approximation on fractals and manifolds is the following:

Theorem 5.1 ([4, Theorem 1.1] (cf. [5, Proposition 3.1]) for fractals, [2, Theorem 1] for manifolds). *Fix $d \in \mathbb{N}$, and let $S \subseteq \mathbb{R}^d$ be either*

- (1) *the limit set of an iterated function system, or*
- (2) *a real-analytic manifold.*

Assume that S is not contained in any proper affine subspace of \mathbb{R}^d . If

$$\operatorname{BA}_d = \{\mathbf{x} \in \mathbb{R}^d : \exists C > 0 \text{ for which (1.1) does not hold for any } \mathbf{p}/q \in \mathbb{Q}^d\},$$

then $\operatorname{BA}_d \cap S$ has full Hausdorff dimension in S .

The set BA_d is called the set of *badly approximable* vectors.

We now claim that Theorem 5.1 also implies an extrinsic analogue of the Jarník–Schmidt theorem. Indeed, the extrinsic analogue of $\text{BA}_d \cap S$ is the set

$$\text{BA}^{\text{ext}} := \{\mathbf{x} \in S : \exists C > 0 \text{ for which (1.1) does not hold for any } \mathbf{p}/q \in \mathbb{Q}^d \setminus S\},$$

and it is a superset of $\text{BA}_d \cap S$. So since $\text{BA}_d \cap S$ has full Hausdorff dimension in S , so does BA^{ext} . This statement is what we refer to as the extrinsic analogue of the Jarník–Schmidt theorem.

Remark 5.2. The extrinsic analogue of the Jarník–Schmidt theorem can be viewed as demonstrating the optimality of Theorem 1.1. Indeed, it demonstrates that for any function $\psi : \mathbb{N} \rightarrow (0, \infty)$ which decays faster than $q \mapsto q^{-(d+1)/d}$, the statement which results from replacing the right hand side of (1.1) by $C\psi(q)$ in Theorem 1.1 cannot be true. See [13] for a detailed discussion of such considerations.

5.2. An analogue of Khinchin’s theorem. When considering analogues of Khinchin’s theorem, we consider only the case of manifolds. The case of fractals is more difficult, since the ambient analogue is not known; moreover, even if it were known, not enough is known about the intrinsic approximation theory of fractals to deduce an extrinsic version from a hypothetical ambient version.

The analogue of Khinchin’s theorem for ambient approximation on manifolds is the following:

Theorem 5.3 ([1, Theorem 2.3]). *Fix $d \geq 2$, and let M be a real-analytic submanifold of \mathbb{R}^d which is not contained in any proper affine subspace of \mathbb{R}^d . Let λ_M denote Lebesgue measure on M . Then for any decreasing function $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$, if the series*

$$(5.1) \quad \sum_{q \in \mathbb{N}} \psi(q)^d$$

diverges,⁶ then the set

$$W_\psi := \left\{ \mathbf{x} \in M : \exists^\infty \mathbf{p}/q \in \mathbb{Q}^d \quad \left\| \mathbf{x} - \frac{\mathbf{p}}{q} \right\| < \frac{\psi(q)}{q} \right\}$$

has full λ_M -measure.

The set W_ψ is called the set of ψ -*approximable* points. Analogously, we define the set of ψ -*intrinsically approximable* and ψ -*extrinsically approximable* points

$$\begin{aligned} W_\psi^{\text{int}} &:= \left\{ \mathbf{x} \in M : \exists^\infty \mathbf{p}/q \in \mathbb{Q}^d \cap M \quad \left\| \mathbf{x} - \frac{\mathbf{p}}{q} \right\| < \frac{\psi(q)}{q} \right\} \\ W_\psi^{\text{ext}} &:= \left\{ \mathbf{x} \in M : \exists^\infty \mathbf{p}/q \in \mathbb{Q}^d \setminus M \quad \left\| \mathbf{x} - \frac{\mathbf{p}}{q} \right\| < \frac{\psi(q)}{q} \right\}. \end{aligned}$$

Obviously, $W_\psi = W_\psi^{\text{int}} \cup W_\psi^{\text{ext}}$. In particular, if W_ψ has full λ_M -measure but W_ψ^{int} has zero λ_M -measure, then W_ψ^{ext} has full λ_M -measure. On the other hand, we have the following:

Theorem 5.4 (Corollary of [11, Theorem 5.5]). *Let M and ψ be as in Theorem 5.3, and suppose that there exist $C, \varepsilon > 0$ such that*

$$(5.2) \quad \psi(q) \leq \frac{C}{q^\varepsilon} \quad \forall q \in \mathbb{N}.$$

Then the set W_ψ^{int} has zero λ_M -measure.

Combining Theorems 5.3 and 5.4, we have the following:

Theorem 5.5. *Let M and ψ be as in Theorem 5.3. Suppose that (5.1) diverges, and also that there exist $C, \varepsilon > 0$ such that (5.2) holds. Then W_ψ^{ext} has full λ_M -measure.*

⁶One may also ask about the converse direction, namely whether the convergence of (5.1) implies that W_ψ has zero λ_M -measure. This is known in some cases; we refer to [1] for details.

Remark 5.6. In this context, the condition (5.2) is quite reasonable. Indeed, by Theorem 1.1, we have

$$(5.3) \quad M = \bigcup_{C>0} W_{C\psi_{1/d}}^{\text{ext}},$$

where $\psi_c(q) := q^{-c}$. Intuitively, this means that the question about the Lebesgue measure of W_ψ^{ext} mostly makes sense if ψ decays faster than $\psi_{1/d}$; if ψ decays more slowly than $\psi_{1/d}$, then (5.3) implies that $W_\psi^{\text{ext}} = M$, and so clearly W_ψ^{ext} has full measure in this case.

5.3. Remarks on the Jarník–Besicovitch theorem. One may ask whether the above techniques can be used to prove an extrinsic analogue of the Jarník–Besicovitch theorem. Again, we consider only the case of manifolds. An analogue of the Jarník–Besicovitch theorem for ambient approximation on manifolds is proven in [1, Theorem 2.5]; we omit the statement for conciseness, although we remark that it only applies to functions ψ which decay more slowly than a fixed function ψ_0 . However, it seems that not enough is known about intrinsic approximation on manifolds to deduce an extrinsic corollary.

To get an idea of what an extrinsic analogue of the Jarník–Besicovitch theorem should look like, we will comment on a well-known example. Let $\dim_H(S)$ denote the Hausdorff dimension of the set S .

Theorem 5.7 ([3, Corollary 2] and [8, Theorem 1]). *Let M be the unit circle in \mathbb{R}^2 . Fix $c > 1/2$, and let $\psi_c(q) = q^{-c}$. Then*

$$(5.4) \quad \dim_H(W_{\psi_c}) = \begin{cases} \frac{2-c}{1+c} & c \leq 1 \\ \frac{1}{1+c} & c \geq 1 \end{cases}.$$

In fact, the “phase transition” which occurs here at $c = 1$ is due to a difference between intrinsic and extrinsic approximation. Specifically, we have the following:

Theorem 5.8. *Let M be the unit circle in \mathbb{R}^2 . Fix $c > 0$, and let $\psi_c(q) = q^{-c}$. Then*

$$(5.5) \quad \dim_H(W_{\psi_c}^{\text{int}}) = \frac{1}{1+c}$$

while

$$(5.6) \quad \dim_H(W_{\psi_c}^{\text{ext}}) = \begin{cases} 1 & c \leq 1/2 \\ \frac{2-c}{1+c} & 1/2 \leq c < 1 \\ 0 & c > 1 \end{cases}.$$

Proof. (5.5) is proven for example in [11, Theorem 2.13]. When $c < 1$, (5.6) follows immediately from (5.4) and (5.5), since $\dim_H(W_{\psi_c}) = \max(\dim_H(W_{\psi_c}^{\text{int}}), \dim_H(W_{\psi_c}^{\text{ext}}))$. Finally, if $c > 1$ then (5.6) is a consequence of [8, Lemma 1]. \square

In particular, from the above example we see that the Hausdorff dimensions of W_{ψ_c} and $W_{\psi_c}^{\text{ext}}$ do not agree if c is large enough. Thus for these values of c , an extrinsic analogue of the Jarník–Besicovitch theorem *could not* be deduced directly from an ambient analogue.

5.4. Open questions.

Open Question 5.9. What is the correct generalization of Proposition 1.5? More precisely, find a class of iterated function systems with the following properties:

- (i) It’s easy to check whether or not any given IFS is in the class.
- (ii) The von Koch snowflake is in the class.
- (iii) No member of the class contains a line segment.

Open Question 5.10. Find an extrinsic analogue of the Jarník–Besicovitch theorem for some class of manifolds or fractals.

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UNIVERSITY OF NORTH TEXAS, DEPARTMENT OF MATHEMATICS, 1155 UNION CIRCLE #311430, DENTON, TX 76203-5017, USA

E-mail address: lior.fishman@unt.edu

OHIO STATE UNIVERSITY, DEPARTMENT OF MATHEMATICS, 231 W. 18TH AVENUE, COLUMBUS, OH 43210-1174, USA

E-mail address: simmons.465@osu.edu

URL: <https://sites.google.com/site/davidsimmonsmath/>